

A NEW INTEGRABLE PROBLEM WITH A QUARTIC INTEGRAL IN THE DYNAMICS OF A RIGID BODY

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Abstract

We consider the problem of motion of a rigid body about a fixed point under the action of an axisymmetric combination of potential and gyroscopic forces. We introduce a new integrable case, valid on zero level of the cyclic integral, that generalizes the known case of motion of a body in liquid due to Chaplygin and its subsequent generalization by Yehia. Apart from certain singular potential terms, the new case involves finite potential and gyroscopic forces, which admit physical interpretation as resulting from interaction of mass, magnetized parts and electric charges on the body with gravitational, electric and magnetic fields.

1 Introduction

Consider a general problem of motion of a rigid body about fixed point O under the action of a combination of conservative potential and gyroscopic forces with a common axis of symmetry, the OZ -axis fixed in space. This problem is described by the Lagrangian

$$L = \frac{1}{2}\omega \mathbf{I} \cdot \omega + \mathbf{l} \cdot \omega - V \quad (1)$$

in which $\mathbf{I} = \text{diag}(A, B, C)$ is the inertia matrix at O , $\omega = (p, q, r)$ is the angular

velocity of the body and $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is the unit vector in the direction of the Z -axis, the scalar and vector potentials V and \mathbf{l} , depend only on $\gamma_1, \gamma_2, \gamma_3$ and all vectors are referred to the body system which we take as the system

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of principal axes of inertia. Equations of motion in terms of the Euler-Poisson variables ω and γ are

$$\dot{\omega}\mathbf{I} + \omega \times (\omega\mathbf{I} + \mu) = \gamma \times \frac{\partial V}{\partial \gamma}, \quad \dot{\gamma} + \omega \times \gamma = \mathbf{0} \quad (2)$$

where

$$\mu = (\mu_1, \mu_2, \mu_3) = \frac{\partial}{\partial \gamma}(\mathbf{l} \cdot \gamma) - \left(\frac{\partial}{\partial \gamma} \cdot \mathbf{l}\right)\gamma \quad (3)$$

Equations (2) admit three general first integrals: Jacobi's integral $I_1 = \frac{1}{2}\omega\mathbf{I} \cdot \omega + V$, the geometric integral $I_2 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$ and the cyclic integral corresponding to cyclic angle of precession around the axis of symmetry of the fields:

$$I_3 = (\omega\mathbf{I} + \mathbf{l}) \cdot \gamma \quad (4)$$

For those equations to be integrable, there should exist a fourth integral I_4 , functionally independent of the above three. A case is named "general" or "conditional" integrable according to whether this complementary integral is valid for arbitrary initial conditions of the motion or only on a single level of the integral I_3 .

The problem (2) was formulated in [1] and was addressed in several subsequent works, mainly from the point of view of integrability. This problem (2) represents a wide generalization of many classical problems in rigid body dynamics that used to be considered separately.

1. **The classical problem of motion of a heavy rigid body about a fixed point** in a uniform gravity field corresponds to the choice $\mathbf{l} = \mu = \mathbf{0}, V = \mathbf{a} \cdot \gamma$ (\mathbf{a} is a constant vector). This is historically the most intensively studied version. For it we have the famous three (and no more) general integrable cases known after Euler, Lagrange and Kowalevski and one conditional integrable case on the zero level of the integral I_3 , bearing the names of Goriachev and Chaplygin (see e.g. [2]).
2. **The problem of motion of a gyrostat**, which is a heavy rigid body with a rotor spinning with a constant angular speed about its axis of symmetry, fixed in the body. The gyrostatic moment can also be due to internal cyclic degrees of freedom such as circulation of fluid in holes inside the body or to forced stationary motions as motors and flow of fluids in circuits in the body (see e.g. [3]). In an interesting alternative, due to Levi-Civita [4], the rotor is left to move freely around its axis of symmetry fixed in the body. In that case the matrix \mathbf{I} is not simply the matrix of inertia of the system, but depends on the direction of the rotor in the body and on the cyclic constant of its motion.

This version corresponds to the choice $\mathbf{l} = \mu = \mathbf{k}$ (\mathbf{k} is a constant vector) $V = \mathbf{a} \cdot \gamma$. For it we have three general integrable cases named after Lagrange, Joukovsky (see e.g. [2]) and Yehia [5], which add the gyrostatic moment to the integrable cases of problem 1. Gavrilov [6] has shown that no more

general integrable cases exist. The conditional case obtained by Sretensky [7] generalizes the above case of problem 1 due to Goriachev and Chaplygin.

3. **The problem of motion of a rigid body by inertia in an ideal incompressible fluid**, infinitely extending and at rest at infinity is traditionally described for a simply connected body by Kirchhoff's equations [8] (see also [9]) or by their Hamiltonian form, mostly used by mathematicians, which are due in their final form to Clebsch [10]. For a perforated body (a body bounded by a multi-connected surface) the equations of motion are usually taken in the form due to Lamb [11], or in the equivalent Hamiltonian form (see e.g. [3]). If one writes those equations in the frame of reference attached to the principal axes of a matrix \mathbf{I} (the inertia matrix of the body modified with an attached distribution of mass that compensates the presence of the fluid), they involve 21 parameters characterizing the shape of the body and, for a perforated body, circulations of the fluid along irreducible contours on its surface. As discussed in [12], the traditional equations of Kirchhoff and Lamb suffer some disadvantages that in most cases lead to a their treatment in isolation from other problems of rigid body dynamics.

The problem under consideration has six degrees of freedom: three for the rotational motion and three for the translation of a point of the body. A new form of the equations of motion of a general body in a liquid was derived by Yehia in [12]. By eliminating the translational motion of the body in terms of the angular velocity and a vector (constant in space), the problem is reduced to one of three degrees of freedom. This form completely fits in the system (2) of a body with one fixed point and corresponds to the choice

$$\begin{aligned} \mathbf{l} &= \mathbf{k} + \gamma \mathbf{K} \\ \mu &= \mathbf{k} + \gamma \bar{\mathbf{K}}, (\bar{\mathbf{K}} = \frac{1}{2} \text{tr}(\mathbf{K}) \delta - \mathbf{K}) \\ V &= \mathbf{a} \cdot \gamma + \frac{1}{2} \gamma \mathbf{J} \cdot \gamma \end{aligned} \tag{5}$$

where \mathbf{K}, \mathbf{J} are constant symmetric 3×3 matrices, δ is 3×3 unit matrix and \mathbf{k}, \mathbf{a} are constant vectors. In this formulation the vectors \mathbf{k}, \mathbf{a} , which result from the circulation of the fluid in the body perforations may be interpreted as a gyrostatic moment and the centre of mass, respectively, of the equivalent body moving about a fixed point.

The equations of motion (2) with μ and V in (5) involve only 18 parameters. The three arbitrary parameters can always be added to the expression for the translational motion of the central point of the body.

For the last problem we know six general integrable cases, a list of which, up to 1986, was provided in [12]. The present number of general integrable cases is still the same. The only change is in the last case (case 6 of [12]),

which is now replaced by a recent case due to Yehia [23]. The latter is a one parameter generalization of an earlier result of Sokolov [13], which, in turn, was a one parameter generalization of Yehia's gyrostat [5].

The list of conditional integrable cases of the present problem is composed of two cases:

- a) A case found by Yehia in [15], which generalizes a classical result of Chaplygin [16] by introducing five parameters and a later result of Goriachev [17] by three parameters.
- b) A case found (with an additional singular potential term) in [18] generalizing the previous Sretensky and Goriachev-Chaplygin cases.

4. The problem of motion of electrically charged rigid body

The potential V can be understood in many cases as due to the scalar interactions of a gravitational field with the mass distribution in the body, an electric field with a permanent distribution of electric charges and a magnetic field with some magnetized parts or steady currents in electric circuits on the body. A constant term \mathbf{k} of the vectors μ and \mathbf{l} is a gyrostatic moment while the variable terms may appear as a result of the Lorentz effect of the magnetic field on the electric charges. For such a model to be realistic one must also assume that the velocity of all mass elements and accelerations of electric charges are sufficiently small to neglect both relativistic effects and classical radiation damping. Let \mathcal{B} and \mathcal{A} be the intensity of the magnetic field and the vector potential of this field at the point \mathbf{r} of the body where the current charge element de is placed. In that case one can write the vector \mathbf{l} as (for details see [1])¹:

$$\mathbf{l} = \mathbf{k} + \int \mathbf{r} \times \mathcal{A} de$$

while μ can be derived from \mathbf{l} according to (3) or constructed directly in the form [1]:

$$\mu = \mathbf{k} - \int (\mathbf{r} \cdot \mathcal{B}) \mathbf{r} de \quad (6)$$

It was noted in [1], that the variable parts of the vector μ appear also in the case of a moving dielectric body in a combination of electric and magnetic fields.

As all the forces acting on the body are supposed to be symmetric about the Z -axis, the combination of static external fields: gravitational field with potential Φ_g , electric field of potential Φ_e and magnetic field \mathcal{B} whose

¹Here MKS units are used. In Gaussian units de should be divided by the velocity of light c (e.g. [14]).

scalar potential is Φ can depend only on Z and $X^2 + Y^2$. The potential of the body can be written as

$$V = \int (\Phi_g dm + \Phi_e de + \mathcal{B} \cdot d\sigma), \quad (7)$$

where dm, de and $d\sigma$ are the mass, electric charge and the magnetic moment contained in the element of the body which at the current moment occupies the point $\mathbf{r}(X, Y, Z)$ of the inertial frame. In most cases we find it is suitable to choose the three potentials Φ_g, Φ_e and Φ to be polynomials in X, Y, Z , subject to Laplace's equation and to the axial symmetry condition. Due to the abundance of physical parameters representing the three distributions and the coefficients of the three potentials, it should be easy to adjust those parameters to match the potential in each case and, moreover, in a variety of choices.

Examples of integrable cases of the present problem are the first four general integrable cases introduced in [20]. In those cases the potential and the components of μ are all polynomials in the Poisson variables $\gamma_1, \gamma_2, \gamma_3$. Two of those cases were given interpretations in the frame of the present problem in [21].

From the mathematical point of view, each of the above problems is a special case of the one coming after it and integrable cases of one problem are usually generalizable to the later ones by means of adding extra-parameters describing additional physical effects. However, it turned out that some integrable potential terms are not likely to be counted for in any of the above four frames. Examples are the potential term $\frac{1}{\gamma_3^2}$ introduced by Goriachev [17] and the term $\frac{1}{\gamma_3^4} - \frac{1}{\gamma_3^6}$ found by Yehia in [23]. Singular terms occur also in several other cases integrable on the zero level of the linear integral I_3 (e.g. [19], [22]). Terms, singular on the plane through the fixed point perpendicular to the axis of symmetry of the fields, cannot be explained as due to electric or magnetic forces, since the latter can have only singular points of the Coulomb type. Thus, certain generalizations of the classical integrable cases are bound to the general problem 2 and not to any of the four physical problems.

In the present note we announce a new conditional case, valid on the level $I_3 = 0$. The new case adds one parameter to the structure of a previous result of Yehia [23]. It is also a 4-parameter generalization of the well-known case of Chaplygin. in the dynamics of a rigid body moving by inertia in an infinitely extended ideal incompressible fluid [16]. Unlike most previous generalizations of integrable cases in rigid body dynamics, the new parameter introduced here evokes finite potential and gyroscopic terms in the equations of motion. This situation makes it possible to construct a physical interpretation of the finite terms as resulting from gravitational, electric and Lorentz interactions.

2 A new integrable problem

The method devised in 1986 by Yehia [24] has proved highly effective for constructing 2D conservative mechanical systems whose configuration spaces are, in general, Riemannian manifolds, which admit a complementary first integral polynomial in the velocity variables. It is suitable for time-reversible systems involving only forces of potential character and also for generalized-natural systems, involving gyroscopic forces as well. For the last systems, the Lagrangian functions include terms linear in velocities.

The culmination of this method for time-reversible systems was the construction of the so-called "master" system [25], which admits a quartic integral and involves the unprecedented number of 21 arbitrary parameters in its structure. This system includes as special cases of it, corresponding to certain values of the parameters, two new integrable cases of rigid body dynamics. Two systems with a cubic integral were found in [28]. One of them is a many-parameter generalization of the two well-known cases of integrable rigid body dynamics, known after Goriachev and Chaplygin [2] and Goriachev [17].

The situation as to time-irreversible systems is much harder. To date, the above mentioned method has produced 41 irreversible cases with a quadratic integral [26], [27]. All relevant known integrable cases of motion of rigid bodies are restored as special cases of some of the new systems and new ones were constructed [27]. The general system with a cubic integral of [28] is generalized into an irreversible system by adding two parameters. This also induced a generalization of the cases of Goriachev and Chaplygin and Goriachev by adding potential and gyroscopic forces, which preserves integrability.

The PDEs that determine gyroscopic generalizations of systems with a quartic integral follow from [24] and are written in detail in [29]. Unlike the case of cubic integrals, they have not been solved for general arbitrary values of parameters in the reversible system. A solution of those equations is found in [29], corresponding to the parameter values that lead only to rigid body dynamics. A new similar version of the same problem is obtained using the same method with a different ansatz of the solution. The resulting Lagrangian system can be written in terms of Euler's angles as generalized coordinates, but the form of the integral is not much tractable. Instead, we present here the final result as an explicit case of integrable Euler-Poisson equations. In this traditional formalism of rigid body dynamics it is easy to check the correctness of the integral and also to compare the present result with the previously known integrable cases of the problem. Thus we formulate the following

Theorem 1 *Let the moments of inertia satisfy the Kowalevski condition $A = B = 2C$ and let the scalar and vector functions V and μ be given by*

$$\begin{aligned} V = & C\{\kappa[2d\gamma_1\gamma_2 + c(\gamma_1^2 - \gamma_2^2)] + \frac{1}{2}n^2\gamma_3^2 - nK\gamma_3[d(\gamma_2^2 - \gamma_1^2) + 2c\gamma_1\gamma_2] \\ & + K^2\left[2cd\gamma_1\gamma_2(\gamma_1^2 - \gamma_2^2) + \frac{d^2}{2}(\gamma_3^4 + 4\gamma_1^2\gamma_2^2) - c^2(\gamma_3^2(\gamma_1^2 + \gamma_2^2) + 2\gamma_1^2\gamma_2^2)\right] \end{aligned}$$

$$+\frac{\lambda}{\gamma_3^2}+\rho(\frac{1}{\gamma_3^4}-\frac{1}{\gamma_3^6})\}. \quad (8)$$

and

$$\mu = C \left(2K\gamma_3(c\gamma_2 - d\gamma_1) - n\gamma_1, \ 2K\gamma_3(d\gamma_2 + c\gamma_1) - n\gamma_2, \ K[d(\gamma_2^2 - \gamma_1^2) + 2c\gamma_1\gamma_2] - 3n\gamma_3 \right) \quad (9)$$

or, equivalently,

$$\mathbf{1} = C(2n\gamma_1, 2n\gamma_2, n\gamma_3 + K[d(\gamma_2^2 - \gamma_1^2) + 2c\gamma_1\gamma_2])$$

where $c, d, n, \lambda, \rho, \kappa$ and K are free parameters, then Euler-Poisson equations of motion (2), which in the present case take the form

$$\begin{aligned} \dot{p} &= \frac{1}{2}qr - \frac{q}{2} [K(d(\gamma_2^2 - \gamma_1^2) + 2c\gamma_1\gamma_2) - 3n\gamma_3] + \frac{r}{2} [2K(c\gamma_1 + d\gamma_2) - n\gamma_2] \\ &\quad + K^2\gamma_3 [cd\gamma_1(3\gamma_2^2 - \gamma_1^2) - d^2\gamma_2(2\gamma_1^2 - \gamma_3^2) + c^2\gamma_2(\gamma_1^2 + \gamma_3^2 - \gamma_2^2)] \\ &\quad + \kappa\gamma_3(c\gamma_2 - d\gamma_1) - \frac{nK}{2} [2c\gamma_1(\gamma_2^2 - \gamma_3^2) - \gamma_2d(2\gamma_3^2 - \gamma_2^2 + \gamma_1^2)] \\ &\quad + \gamma_2 \left(\frac{n^2\gamma_3}{2} - \frac{\lambda}{\gamma_3^3} - \frac{2\gamma_3^2 - 3}{\gamma_3^7} \rho \right), \\ \dot{q} &= -\frac{pr}{2} + \frac{p}{2} [K(d(\gamma_2^2 - \gamma_1^2) + 2c\gamma_1\gamma_2) - \frac{r}{2} [2K(c\gamma_2 - d\gamma_1) - n\gamma_1] \\ &\quad - K^2\gamma_3 [cd\gamma_2(3\gamma_1^2 - \gamma_2^2) - d^2\gamma_1(2\gamma_2^2 - \gamma_3^2) + c^2\gamma_1(\gamma_2^2 + \gamma_3^2 - \gamma_1^2)] \\ &\quad + \kappa\gamma_3(c\gamma_1 + d\gamma_2) + \frac{nK}{2} [2c\gamma_2(\gamma_1^2 - \gamma_3^2) - d\gamma_1(\gamma_1^2 - \gamma_2^2 - 2\gamma_3^2)] \\ &\quad + \gamma_1 \left(\frac{2\gamma_3^2 - 3}{\gamma_3^7} \rho + \frac{\lambda}{\gamma_3^3} - \frac{n^2}{2} \gamma_3 \right), \\ \dot{r} &= [2K(c\gamma_2 - d\gamma_1)\gamma_3 - n\gamma_1] q - [2K(c\gamma_1 + d\gamma_2)\gamma_3 - n\gamma_2] p \\ &\quad + 2K^2 [cd(\gamma_1^4 + \gamma_2^4 - 6\gamma_1^2\gamma_2^2) - 2(c^2 - d^2)\gamma_1\gamma_2(\gamma_1^2 - \gamma_2^2)] \\ &\quad - 2\kappa[2c\gamma_1\gamma_2 + d(\gamma_2^2 - \gamma_1^2)] - 2nK\gamma_3 [c(\gamma_1^2 - \gamma_2^2) + 2d\gamma_1\gamma_2], \\ \dot{\gamma}_1 &= r\gamma_2 - q\gamma_3 \\ \dot{\gamma}_2 &= p\gamma_3 - r\gamma_1 \\ \dot{\gamma}_3 &= q\gamma_1 - p\gamma_2 \end{aligned}$$

are integrable on the zero level of the integral

$$I_3 = 2p\gamma_1 + 2q\gamma_2 + \{r + K[d(\gamma_2^2 - \gamma_1^2) + 2c\gamma_1\gamma_2]\}\gamma_3 - n\gamma_3^2$$

The complementary integral of the motion is

$$\begin{aligned} I_4 &= \left[(p + n\gamma_1)^2 - (q + n\gamma_2)^2 + c\kappa\gamma_3^2 + \gamma_3^2 (Kd(r + n\gamma_3) + cK^2(c(\gamma_1^2 - \gamma_2^2) + 2d\gamma_1\gamma_2) - \frac{\lambda(\gamma_1^2 - \gamma_2^2)}{\gamma_3^2}) \right]^2 \\ &\quad + \left[2(p + n\gamma_1)(q + n\gamma_2) + d\kappa\gamma_3^2 + (dK^2(c(\gamma_1^2 - \gamma_2^2) + 2d\gamma_1\gamma_2) - Kc(r + n\gamma_3))\gamma_3^2 - \frac{2\lambda\gamma_1\gamma_2}{\gamma_3^2} \right]^2 \end{aligned}$$

$$\begin{aligned}
& +\rho\left\{\frac{2(\gamma_3^2-1)}{\gamma_3^6}[(p+n\gamma_1)^2+(q+n\gamma_2)^2]-\frac{2K(r+n\gamma_3)[2c\gamma_1\gamma_2+d(\gamma_2^2-\gamma_1^2)]}{\gamma_3^4}\right. \\
& +\frac{(1-\gamma_3^2)^2}{\gamma_3^{12}}(\rho-2\lambda\gamma_3^4)+K^2[2c^2(\frac{1}{\gamma_3^4}-\frac{2}{\gamma_3^2})+8\frac{(d^2-c^2)\gamma_1^2\gamma_2^2+cd\gamma_1\gamma_2(\gamma_1^2-\gamma_2^2)}{\gamma_3^4}] \\
& \left.+\frac{2\kappa}{\gamma_3^4}[c(\gamma_1^2-\gamma_2^2)+2d\gamma_1\gamma_2]\right\}.
\end{aligned} \tag{10}$$

This case, involving 7 significant parameters, is a new generalization of the previously known integrable problems in rigid body dynamics:

Author- year	Conditions on parameters
Yehia [23] (§4.2.3) 2003	$K = 0$
Goriachev[17] 1916	$K = n = \rho = 0$
Chaplygin [16] 1903	$K = n = \rho = \lambda = 0$

3 Physical interpretation:

The Goriachev parameter λ and Yehia's parameter ρ give rise to a singular plane of the potential. That is the plane $\gamma_3 = 0$, orthogonal to the space axis of symmetry of the fields applied to the body. For a physical interpretation we set

$$\lambda = \rho = 0 \tag{11}$$

The potential becomes a quartic polynomial in the Poisson variables containing cubic and quadratic terms. As all the forces acting on the body are symmetric about Z - axis. Let there be the following combination of static external fields: a gravitational field with potential Φ_g , an electric field of potential Φ_e and magnetic field \mathcal{B} whose scalar potential is Φ . Note that the three potentials can depend only on Z and $X^2 + Y^2$. The potential of the body can be written in the form (7), which in the present case is a polynomial expression of degree 4 in the components of γ . Thus, in this case we find that it suffices to choose the three potentials Φ_g , Φ_e and Φ to be polynomial solutions of Laplace's equation subject only to the axial symmetry condition. The potential V can be realized in a variety of ways.

The vector μ can be expressed directly in terms of the magnetic field according to the formula (6).

In the case when the scalar potential of the external magnetic field can be expressed as a second-degree harmonic polynomial

$$\Phi = a_1 Z + a_2(3Z^2 - r^2) \tag{12}$$

The vector μ can be expressed in terms of the body system of coordinates,

$$\mu = \int [a_1 \mathbf{r} \cdot \gamma + 2a_2(3(\mathbf{r} \cdot \gamma)^2 - r^2)] \mathbf{r} d\epsilon \tag{13}$$

Finally, we can write

$$\begin{aligned}
\mu_1 &= -2a_2(I_{xxx} + I_{xyy} + I_{xzz}) + a_1(I_{xx}\gamma_1 + I_{xy}\gamma_2 + I_{xz}\gamma_3) \\
&\quad + 6a_2(I_{xxx}\gamma_1^2 + I_{xyy}\gamma_2^2 + I_{xzz}\gamma_3^2 + 2I_{xxy}\gamma_1\gamma_2 + 2I_{xxz}\gamma_1\gamma_3 + 2I_{xyx}\gamma_2\gamma_3) \\
\mu_2 &= -2a_2(I_{xxy} + I_{yyx} + I_{yzz}) + a_1(I_{xy}\gamma_1 + I_{yy}\gamma_2 + I_{yz}\gamma_3) \\
&\quad + 6a_2(I_{xxy}\gamma_1^2 + I_{yyx}\gamma_2^2 + I_{yzz}\gamma_3^2 + 2I_{xyx}\gamma_1\gamma_2 + 2I_{xyx}\gamma_1\gamma_3 + 2I_{yyz}\gamma_2\gamma_3) \\
\mu_3 &= -2a_2(I_{xxz} + I_{yyz} + I_{zzz}) + a_1(I_{xz}\gamma_1 + I_{yz}\gamma_2 + I_{zz}\gamma_3) \\
&\quad + 6a_2(I_{xxz}\gamma_1^2 + I_{yyz}\gamma_2^2 + I_{zzz}\gamma_3^2 + 2I_{xyz}\gamma_1\gamma_2 + 2I_{xxz}\gamma_1\gamma_3 + 2I_{yzz}\gamma_2\gamma_3)
\end{aligned} \tag{14}$$

where, for example, $I_{xx} = \int x^2 de$, $I_{xyz} = \int xyz de$ and so forth are the second- and third-degrees moments of the charge distribution.

It is not hard now to verify that the finite-potential version the new integrable case, subject to the conditions (11), corresponds to the choice

$$\begin{aligned}
I_{xy} &= I_{xz} = I_{yz} = I_{xxx} = I_{xxy} = I_{xyy} = I_{xxz} = I_{yyx} = I_{yzz} = I_{zzx} = I_{xzz} + I_{yyz} = 0 \\
I_{yy} &= I_{xx} = \frac{I_{zz}}{3} = -\frac{Cn}{a_1}, 6a_2I_{xxz} = -CKd, 6a_2I_{xyz} = cCK
\end{aligned} \tag{15}$$

This is a set of conditions on the second and third moments of the charge distribution. Regarding the fact that the charge distribution is not necessarily

positive, no restrictions like the triangle inequalities on inertia moments are necessary here. It turns out that the charge distribution can be determined with great arbitrariness.

References

- [1] Yehia H M 1986 On the motion of a rigid body acted upon by potential and gyroscopic forces: I. The equations of motion and their transformations J. Mécan. Théor. Appl. 5 747–54.
- [2] Leimanis E 1965 The General Problem of Motion of Coupled Rigid Bodies About a Fixed Point. Berlin: Springer.
- [3] Borisov A.V., Mamaev I.S. 2005 Rigid body dynamics - Hamiltonian methods, integrability, chaos. Moscow - Izhevsk: Institute of Computer Science, 576 p (In Russian)
- [4] Levi-Civita T. and Amaldi U., Lezioni di meccanica razionale. Bologna : Zanichelli, 1950
- [5] Yehia H. M. 1986 New integrable cases in the dynamics of rigid bodies. Mech. Res. Commun., 13, 3, 169-172

- [6] Gavrilov L., Non-integrability of the equations of heavy gyrost. Compositio Math. 82, 3, 275-291 (1992).
- [7] Sretensky LN 1963 On certain cases of motion of a rigid body with a gyroscope. Vestnik Moskov Univ. Ser. 1. Mat. Mekh., 3, 60-71.
- [8] Kirchhoff G.R. Über die Bewegung eines Körpers in einer Flüssigkeit. J. Reine Angew. Math., 1870.
- [9] Kirchhoff G.R. Vorlesungen über mathematische Physik. Mechanik. Leipzig, 1874.
- [10] Clebsch A. Über die Bewegung eines Körpers in einer Flüssigkeit. Math. Annalen, 1871, Bd. 3, S. 238-262.
- [11] Lamb H. Hydrodynamics, ed. 6-th. N. Y., Dover publ., 1945
- [12] Yehia H. M. (1986) On the motion of a rigid body acted upon by potential and gyroscopic forces: II: A new form of the equations of motion of a multiconnected rigid body in an ideal incompressible fluid. J. Mecan. Theor. Appl., 5, 755-762.
- [13] Sokolov V.V., 2002 A generalized Kowalevski Hamiltonian and new integrable cases on $e(3)$ and $so(4)$. In "Kowalevski property", ed. V. B.Kuznetsov, CRM Proceedings and Lect. Notes, AMS, p. 307-315.
- [14] Bradbery T C 1968 Theoretical Mechanics (New York: Wiley)
- [15] Yehia H. M., 1987 New integrable cases in the dynamics of rigid bodies III. Mech. Res. Commun., 14, 3, 177-180.
- [16] Chaplygin S. A. (1903), A new partial solution of the problem of motion of a rigid body in a liquid. Trudy Otdel. Fiz. Nauk Obsh. Lyub. Est., 117-110.
- [17] Goriatchev D. N., New case of integrability of the Euler dynamical equations. Varshav. Univ. Izvest., No 3, 1-13 (1916).
- [18] Sokolov V.V. and Tsiganov A.V. 2002 Lax Pairs for the Deformed Kowalevski and Goryachev-Chaplygin Tops. Theor. Math. Phys. 131, 1, 543-549.
- [19] Yehia H. M., Two-dimensional conservative mechanical systems with quartic second integral. Reg. Chaot. Dyn., 11, 103 - 122 (2006).
- [20] Yehia H. M., New generalizations of the integrable problems in rigid body dynamics. J. Phys A: Math. Gen. 30, 7269 - 7275 (1997).
- [21] Yehia H. M., New generalizations of all the known integrable problems in rigid body dynamics. J. Phys. A: Math. Gen., 32, 7565 - 7580 (1999).

- [22] Yehia H. M. and Elmandouh A. A. (2011), New conditional integrable cases of motion of a rigid body with Kowalevski's configuration. *J. Phys. A: Math. Theor.* 44 012001.
- [23] Yehia H. M. (2003), Kowalevski's integrable case: Generalizations and related new results. *Regul. Chaotic. Dyn.* 8 337-348.
- [24] Yehia H. M. (1986), On the integrability of certain problems in particle and rigid body dynamics. *J. Mecan. Theor. Appl.*, 5, 1, 55-71.
- [25] Yehia H. M. (2006), The Master integrable two-dimensional system with a quartic second integral. *J. Phys. A: Math. Gen.*, 39, 5807 - 5824.
- [26] Yehia H. M., Generalized natural mechanical systems of two degrees of freedom with quadratic integrals. *J. Phys A: Math. Gen.*, 25, 197-221 (1992).
- [27] Yehia H. M. (2007), Atlas Of Two-Dimensional Irreversible Conservative Lagrangian Mechanical Systems With A Second Quadratic Integral. *J. Math. Phys.*, 48, 082902.
- [28] Yehia H. M. (2002), On certain two-dimensional conservative mechanical systems with cubic second integral. *J. Phys. A: Math. Gen.*, 35, 9469-9487.
- [29] Yehia H. M. and Elmandouh A. A. (2008), New Integrable Systems with a Quartic Integral and New Generalizations of Kowalevski's and Goriatchev's Cases. *Reg. Chaotic Dyn.*, Vol. 13, 57-69.